Understanding earthquakes by the tsunami waves they cause

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Problem Setup

In our project, we consider a tsunami wave as it comes into a long stretch of beach, such as the one seen in Figure $1(a)$. As the tsunami wave approaches the coast the shoreline moves up and down the beach, motion which is called run-up or draw-down. In our project we develop a mathematical model which describes the run-up and draw-down of the tsunami wave (see in part Figure $1(b)$). Using this model and knowledge about the movement of the shoreline we're able to recover information about the earthquake that generated tsunami wave. In particular, we will show how to determine the displacement of the wave caused by the earthquake. In the following sections we setup the mathematical model to approach this problem, however a visual of the variables of interest can be seen in Figure $1(b)$.

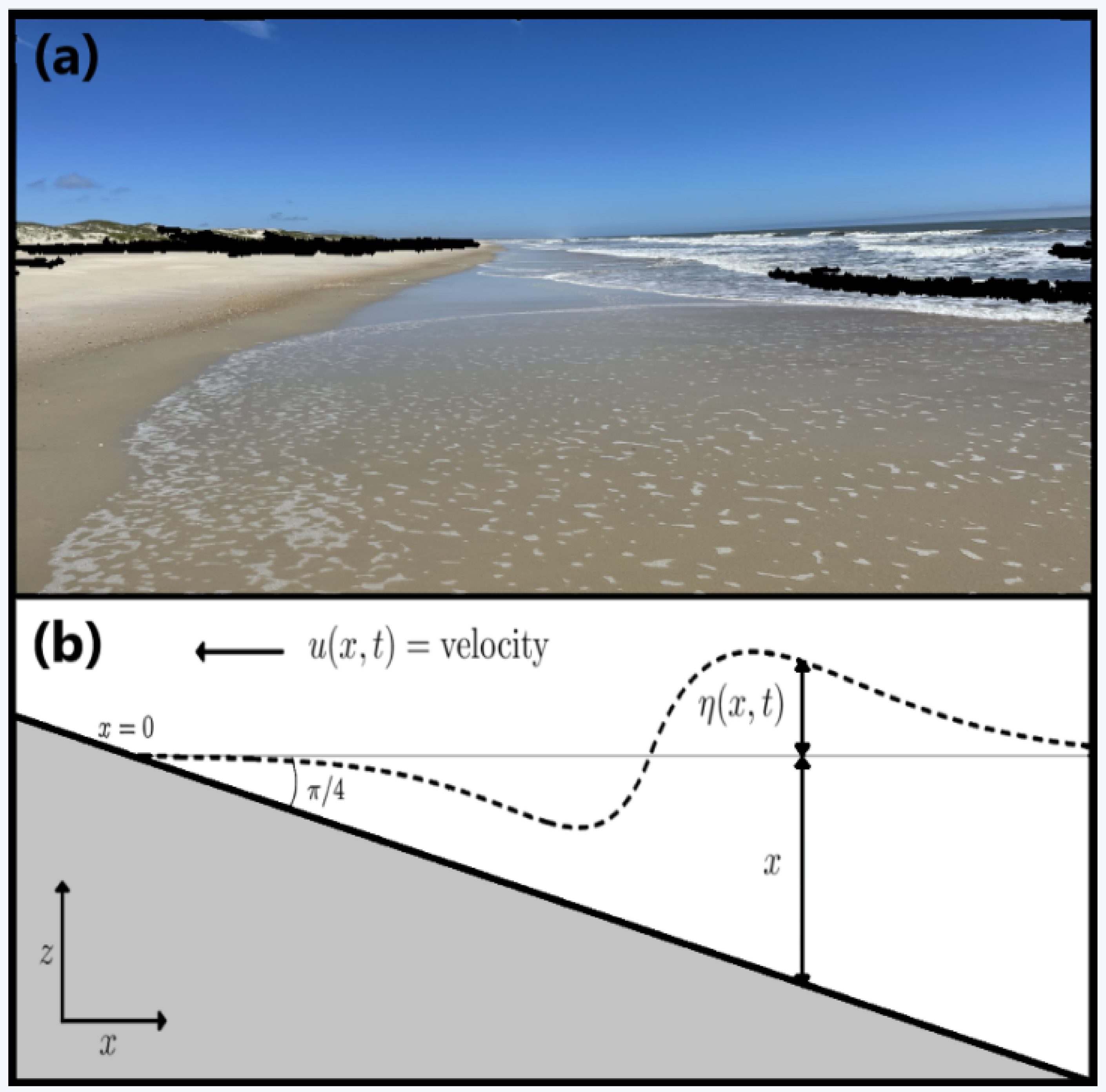

Figure 1.(a) A plane beach like the one we're concerned with in our model, (b) a visual description of the mathematical model and the variables involved used in our problem.

Mathematical Model

In the analysis of tsunami waves the one-dimensional shallow water wave equations in (1) are extensively used (see [3],[4]).

$$\begin{aligned}&\tfrac{\partial}{\partial t}\eta + \tfrac{\partial}{\partial x}[(x+\eta)u] = 0\\ &\tfrac{\partial}{\partial t}u + u\tfrac{\partial}{\partial x}u + \tfrac{\partial}{\partial x}\eta = 0\end{aligned} \qquad (1)$$

These equations, with initial conditions $\eta(x,0) = \eta_0(x)$ and $u(x,0) = 0$ form the starting point of our model. Using the Carrier-Greenspan transform:

$$\begin{aligned}&\varphi(\sigma,\tau) = u(x,t),\ \psi(\sigma,\tau) = \eta(x,t) + u^2(x,t)/2\\ &\sigma^2 = x + \eta(x,t),\ \ \tau = (t - u(x,t))/2\end{aligned}. \qquad (2)$$

equation (1) turns into the wave equation

$$\frac{\partial^2}{\partial \tau^2}\psi = \frac{\partial^2}{\partial \sigma^2}\psi + \sigma^{-1}\frac{\partial}{\partial \sigma}\psi$$

which has the advantage of fixing the shoreline at the line $\sigma = 0$, a useful fact for our derivation.

Shoreline Equation

Using Hankel transforms, we derive the following equation

$$\psi(0,\tau) = \frac{d}{d\tau}\operatorname{sgn}\tau\left[\int_0^{|\tau|}\frac{s\psi_0(s)}{\sqrt{\tau^2 - s^2}}dx\right].$$

This equation describes ψ at the location of the moving shoreline in terms of a convolution. Now, using the fact that convolutions are invertible we arrive at the following equation (the shoreline equation)

$$\psi_0(\sigma) = \frac{2}{\pi}\int_0^{\sigma}\frac{\Psi(s)ds}{\sqrt{\sigma^2 - s^2}}, \qquad (3)$$

where $\Psi(\tau) = \psi(0,\tau)$. The shoreline equation is notable, due to the isolation $\psi_0(\sigma)$ our initial conditions and provides the basis for the inverse problem.

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Inverse Problem

If we know the motion of the shoreline, given by $x_0(t)$, the inverse problem can now be solved as follows:

1. Compute $\Psi(\tau)$ implicitly using
   $$\psi(\tau) = -x_0(t) + v_0(t)^2/2,\ \ \tau = (t - v_0(t))/2,$$
   where $v_0(t) = dx_0(t)/dt$ is the shoreline velocity.
2. Then compute $\psi_0(\sigma)$ from (3).
3. From (2) use
   $$\eta_0(x) = \psi_0(\sigma),\ \ \sigma = \sqrt{\eta_0(x) + x}$$
   to solve implicitly for the initial displacement η_0.

Conclusions

We have established a well-working algorithm for the recovery of the initial conditions of the tsunami wave from data collected at the shoreline. In the future, we wish to extend our work here to more complicated cases such as for zero initial displacement and nonzero initial velocity, as well as other physically applicable cases. Additionally, consideration should be given to more complicated bathymetries, like the U and V-shaped bays considered in [2] and [3] or piecewise sloping bays like those analyzed in [1].

Literature Cited

References

[1] Anderson, D., Harris, M., Hartle, H., Nicolsky, D., Pelinovsky, E., Raz, A., and Rybkin, A. Run-Up of Long Waves in Piecewise Sloping U-Shaped Bays. Pure Appl. Geophys. 174 (2017), no. 8, 3185–3207. https://doi.org/10.1007/s00024-017-1476-3.

[2] Hartle, H., Rybkin, A., Pelinovsky, E., and Nicolsky, D. Robust computations of runup in inclined U- and V-shaped bays. Pure Appl. Geophys. 178(12) (2021), 5017–5029. https://doi.org/10.1007/s00024-021-02877-x.

[3] Nicolsky, D., Pelinovsky, E., Raz, A., and Rybkin, A. General initial value problem for the nonlinear shallow water equations: Runup of long waves on sloping beaches and bays. Physics Letters A, 382 (38) (2018), 2738–2743.

[4] Rybkin, A., Nicolsky, D., Pelinovsky, E., Buckel, M. The generalized Carrier-Greenspan transform for the shallow water system with arbitrary initial and boundary conditions. Water Waves 3, 267–296 (2021). https://doi.org/10.1007/s42286-020-00042-w.