<u>**SAMPLE**</u> <u>**SAMPLE**</u>

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Numerical Linear Algebra Comprehensive Exam

Part I: do **ALL** of the problems **A**–**F**

A. Suppose $A \in \mathbb{C}^{m\times m}$ is invertible. Consider the factorization $PA = LU$ for L unit lower-triangular, U upper-triangular, and P a permutation matrix.

(a) Given $b \in \mathbb{C}^m$, how is this factorization used to solve $Ax = b$ for $x \in \mathbb{C}^m$?

(b) Give leading order estimates, as $m \to \infty$, of the number of floating point operations to implement the major stages of the method in part **(a)**, including the cost of the factorization $PA = LU$. Assume the factorization is computed by Gaussian elimination with partial pivoting and that the other major steps use standard algorithms. (*Name these standard algorithms.*)

(c) Is Gaussian elimination with partial pivoting always the best method for solving a square system like $Ax = b$? Describe at least one alternative algorithm with superior stability properties.

B. **(a)** Suppose $A \in \mathbb{C}^{m\times n}$. Define a *singular value decomposition* (SVD) of A. For $m > n$, describe how the *reduced* SVD differs from the SVD.

(b) For $m > n$ and A of full rank, use the reduced SVD to construct an orthogonal projector P onto the range of A. Show that P is an orthogonal projector.

C. **(a)** Compute $\|A\|_2$ and $\|A\|_F$ for the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

(b) Compute the condition number $\text{cond}(A) = \kappa(A)$ using the 2-norm.

D. Let $\|\cdot\|$ be any norm on $\mathbb{C}^m$ and also let it denote the corresponding induced norm on square matrices $A \in \mathbb{C}^{m\times m}$. Show that $\rho(A) \le \|A\|$ if $\rho(A)$ is the spectral radius of A. (*The* spectral radius *of A is the maximum of $|\lambda|$ over the eigenvalues λ of A.*)

E. Consider a computer satisfying the standard axioms for floating point arithmetic,[1] so that the machine precision ϵ_{mach} is precisely defined. Let X, Y be real, normed vector spaces and let $f : X \to Y$ be a problem. Precisely define: The algorithm $\tilde{f} : X \to Y$ computing the problem f is *backward stable*. (An informal definition might be included to explain the idea, but it does not suffice.)

F. Give an example of an invertible 2×2 matrix A which has $\det(A) > 10^{20}$ but for which the system $Ax = b$ is well-conditioned.

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[1]Namely that for each such computer there exists $\epsilon_{\text{mach}} > 0$ so that the following two facts hold: (1) For all $x \in \mathbb{R}$ there is ϵ so that $|\epsilon| \le \epsilon_{\text{mach}}$ and $fl(x) = x(1 + \epsilon)$. (2) For all $x, y \in \mathbb{R}$ and each operation $* = +, -, \times, \div$, with computer implementation $\circledast$, there is ϵ so that $|\epsilon| \le \epsilon_{\text{mach}}$ and $x \circledast y = (x * y)(1 + \epsilon)$.

Part II: do **TWO** of the following three problems

1. **(a)** Let

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}.$$

Compute the eigenvalues and eigenvectors of A.

(b) Let $x \in \mathbb{C}^2$ be a random vector. For the same matrix A, estimate

$$\frac{\|A^{2013}x\|}{\|A^{2012}x\|}.$$

Explain. Include in your explanation a description of those rare vectors for which your estimate is not accurate.

2. Suppose we define a square matrix $A \in \mathbb{C}^{m\times m}$ to be *normal* if there is an orthonormal basis of $\mathbb{C}^m$ consisting of eigenvectors of A.

(a) Show that if A is normal then $A^*A = AA^*$.

(b) Show that if A is normal then any Schur factorization of A is, in fact, a unitary diagonalization.

3. **(a)** Show that if P is an orthogonal projector then $I - 2P$ is unitary.

(b) For $A \in \mathbb{C}^{m\times n}$ of full rank with $m \geq n$, A^* the hermitian transpose of A, $b \in \mathbb{C}^m$, and P the orthogonal projector onto the range of A, show that the equations

$$A^*Ax = A^*b \qquad \text{and} \qquad Ax = Pb$$

have the same unique solution $x \in \mathbb{C}^n$.