SAMPLE

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SAMPLE [AUTHOR: Maxwell/Bueler]

Real Analysis Comprehensive Exam

Complete **EIGHT** of the following ten problems. It is better to fully complete fewer problems than to earn partial credit on many problems.

1. Compute, with justification,

$$\lim_{n \to \infty} \int_0^1 \sqrt{\frac{nx}{1 + n^2 x^2}} \, dx.$$

- 2. Give examples of the following
 - (a) A sequence (f_n) in C[0,1] that converges pointwise to 0 but such that $\int_0^1 f_n \neq 0$.
 - (b) A bounded sequence in ℓ_1 that has no convergent subsequence.
 - (c) A sequence in C[0, 1] that converges pointwise to a discontinuous function.
- 3. Let (f_n) be a bounded sequence of functions in C[0, 1]. Define

$$F_n(x) = \int_0^x f_n(s) \, ds.$$

Show that (F_n) has a uniformly convergent subsequence.

- 4. (a) State the Axiom of Completeness for \mathbb{R} .
 - (b) Prove that a monotone increasing sequence of real numbers converges if and only if it is bounded above.
- 5. Let m^* denote Lebesgue outer measure. Suppose that E is a subset of \mathbb{R} such that $m^*(E \cap (a, b)) \leq 3/4(b-a)$ for every interval (a, b). Prove that $m^*(E) = 0$.
- 6. Let (a_n) be a bounded sequence. Show that $\sum_{n=1}^{\infty} a_n e^{-nx}$ defines a continuous function on [1, 2].
- 7. Suppose that X is compact and $f : X \to \mathbb{R}$ is continuous. Prove that f is uniformly continuous.
- 8. Suppose $f \in L^1(\mathbb{R})$ is uniformly continuous. Show that $\lim_{x\to\infty} f(x) = 0$.
- 9. Let $f \in L^1[-\pi,\pi]$. Show that

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx = 0.$$

Hint: First consider the case where f is the characteristic function of an interval.

10. Let $f : [0,1] \to \mathbb{R}$ be an increasing, but not necessarily continuous function. Show that f is Borel measurable.