Sample Algebra Comprehensive Exam Problems

Spring 2019

This sample exam is a composite of questions asked on recent MATH 631 exams. As a result the length of this exam is much longer (by a factor of at least 2) than a comprehensive exam, but the breadth of topics covered and the difficulty level are representative of possible test questions.

Part I. Short answer or easy computation.

1. Consider the cylic group $C_{4900} = \langle x \rangle$ of order $4900 = 2^2 \cdot 5^2 \cdot 7^2$.

   (a) Give the number of generators of C_{4900}.

   (b) List explicitly the elements x^a, with $0 \le a \le 4899$, of order 10.
   Answer: $|x^a| = 10$ if $a =$ ____.

   (If it helps, you can simply give the prime factorizations of a. I am not interested in your ability to multiply integers.)

2. Consider the cyclic groups $\mathbb{Z}/30\mathbb{Z}$ and $C_{18} = \langle x \rangle$ of orders 30 and 18 respectively, and suppose that

$$\begin{aligned} \varphi_a : \mathbb{Z}/30\mathbb{Z} &\to C_{18} \\ 1 &\mapsto x^a \end{aligned}$$

   extends to a well-defined group homomorphism from $\mathbb{Z}/30\mathbb{Z}$ to C_{18}.

   (a) List the values of a with $0 \le a \le 17$ for which this is true. (I.e. The map defines a well-defined group homomorphism.)

   (b) Give a brief explanation why such a well-defined group homomorphism can not be surjective.

3. Consider the symmetric group $G = S_7$ and let $\sigma = (1\ 2\ 3\ 6\ 5\ 4\ 7)$ be a 7-cycle.

   (a) Express σ as the product of (not necessarily disjoint) transpositions.

   (b) Compute the number of conjugates of σ in S_7.

   (c) Let τ be the 7-cycle $(3\ 7\ 1\ 4\ 5\ 6\ 2)$. Give an element α that conjugates σ to τ, i.e. give α such that $\alpha\sigma\alpha^{-1} = \tau$.

   (d) Noting that S_7 acts on itself by conjugation, explicitly use the Orbit-Stabilizer theorem to find the size of the stabilizer of σ under this action and the elements of the Stabilizer subgroup of S_7.
   The stabilizer of σ in this context is better known as ________________. (Using appropriate notation in place of words here is fine.)

   (e) Noting that $\sigma \in A_7$, what is the size of the conjugacy class of σ in A_7? Stated otherwise, how many conjugates in A_7 does σ have? Briefly, state a result that justifies your answer.
   Answer: The number of conjugates of σ in A_7 is ___________

   because ....

4. (a) Suppose that A is an Abelian group of order $200 = 2^3 \cdot 5^2$. Give the isomorphism classes for A in a table below. In the left hand column, give the elementary divisor decomposition and in the right hand column, give the invariant factor decomposition. **Groups on the same row should be isomorphic.** You do not need to show your work.

   (b) Give the number of non-isomorphic Abelian groups of order $400 = 2^4 \cdot 5^2$.

5. Prove that there are no simple groups of order 56.

6. Give the definition of a nilpotent element in a ring R. Then prove that the set of nilpotent elements in $M_2(\mathbb{Q})$ is **not** an ideal.

7. Suppose G is a non-cyclic group of order $205 = 5 \cdot 41$. Give, with proof, the number of elements of order 5 in G.

8. Find **ALL** solutions x in the integers to the simultaneous congruences.

$$\begin{aligned} x &\equiv 7 \mod 11 \\ x &\equiv 2 \mod 5 \end{aligned}$$

9. Draw the lattice diagram of prime ideals for the polynomial ring $\mathbb{Q}[x]$. *Note:* There are infinitely many prime ideals so you will need a way to indicate them all.

Part II. Theory

1. Suppose G is a group with H, K subgroups of G. Prove that if $H \leq N_G(K)$, then $HK = \{hk \mid h \in H, k \in K\}$ is a subgroup of G.

2. Suppose that a finite group G is of order 105, $|G| = 3 \cdot 5 \cdot 7$, and that G has normal subgroups of order 3, 5 and 7. Prove or disprove: G is cyclic.

3. Let P be a p-group, $|P| = p^a > 1$ for p a prime, and let A be a nonempty finite set. Suppose that P acts on A and define *the set of fixed points* of this action:

$$A_0 = \{a \in A \mid g \cdot a = a \text{ for every } g \in P\}.$$

Prove that

$$|A| \equiv |A_0| \pmod{p}.$$

4. Let $\varphi(n)$ denote the Euler φ-function. Prove that if p is a prime and $n \in \mathbb{Z}^+$, then

$$n \,\Big|\, \varphi(p^n - 1).$$

(Hint: Compute the order of $\bar{p}$ in the appropriate group first.)

5. Prove that if G is a group of order p^2 for p a prime, then G is Abelian.

6. Suppose G is a finite group of order $|G| = 14,553 = 3^3 \cdot 7^2 \cdot 11$ and that N is a normal subgroup of G of order $|N| = 11$. Prove that $N \leq Z(G)$.

7. Suppose G is a group, $H \leq G$, and $\mathrm{Aut}(H)$ the group of automorphisms of H.
   (a) Using the First Isomorphism theorem, give a **full** proof of the following statement.
   The quotient group $N_G(H)/C_G(H) \cong A \leq \mathrm{Aut}(H)$.
   (b) Suppose now that P is a Sylow p-subgroup of S_p for a prime p. Prove that

$$N_{S_p}(P)/C_{S_p}(P) \cong \mathrm{Aut}(P).$$

8. Let G be a finite group of order 22. Prove that G is cyclic or isomorphic to the dihedral group D_{22}.

9. In a PID every nonzero element is a prime if, and only if, it is irreducible.

10. Suppose R is a commutative ring with 1 and for each $x \in R$, there is a positive integer $n > 1$ so that $x^n = x$. Prove that every nonzero prime ideal is maximal.

11. Let $\mathbb{F}_7$ denote the finite field with 7 elements.
    (a) Explicitly construct a finite field with $343 = 7^3$ elements. Explain your work.
    (b) The field you constructed in part (a) is a simple extension of $\mathbb{F}_7$ so let α be an element in some extension of $\mathbb{F}_7$ such that $|\mathbb{F}_7(\alpha)| = 343$. Find the inverse of the element $1 + \alpha \in \mathbb{F}_7(\alpha)$.

12. Find, with brief justification, all ring homomorphisms from $\mathbb{Z} \to \mathbb{Z}/12Z$.

13. Consider the ring of Gaussian integers $\mathbb{Z}[i]$.

    (a) Prove that if $\alpha = a + bi$ for $a, b \in \mathbb{Z}$ is a Gaussian integer with $N(\alpha) = p$ for p a prime of $\mathbb{Z}$, then α is irreducible.

    (b) List all the units of $\mathbb{Z}[i]$.

    (c) Give an example of a prime number $p \in \mathbb{Z}$ such that p is irreducible in $\mathbb{Z}[i]$. Justify your answer by stating an appropriate result.

14. Let D be a square-free integer, and consider the quadratic number field $\mathbb{Q}(\sqrt{D})$ and its subring of integers $\mathcal{O}$. Let $N : \mathbb{Q}(\sqrt{D}) \to \mathbb{Z}$ denote the field norm map which is multiplicative. The restriction of N to the ring of integers $\mathcal{O}$ will also denoted by N.

    (a) Prove that an element $\alpha \in \mathcal{O}$ is a unit if, and only if, $N(\alpha) = \pm 1$.

    (b) When $D = -3$, the ring of integers is $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\left(\frac{1+\sqrt{-3}}{2}\right)$. Find a unit in $\mathcal{O} \setminus \mathbb{Z}$.

    (c) Let $D = -5$. Give, with proof, an example of an element $x = a + b\sqrt{-5}$ for $a, b \in Z$ such that x is irreducible, but x is not prime in $\mathbb{Z}[\sqrt{-5}]$.