SAMPLE SAMPLE

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Comprehensive MS Math (and PhD qualifying) Exam:

Complex Analysis

Rules. Complete as many problems as you can. (It may be a good idea to use the last 10 minutes to write down *what you intend to do* on the problems on which you are otherwise stuck.)

1. Show that: a) $\operatorname{Re} z\bar{w} = \operatorname{Re} \bar{z}w$ and $\operatorname{Im} z\bar{w} = -\operatorname{Im} \bar{z}w$.
b) if $z + w$ and zw are both real then either z and w are both real or else $z = \bar{w}$.

2. Find a convergent series $\sum_{k=0}^{\infty} z_k$, with $z_k \neq 0$, such that $\limsup |z_{k+1}/z_k| = \infty$.

3. Find the radius of convergence of the series $\sum_{k=0}^{\infty} a_k z^k$ with $a_{3n} = 1/n$, $a_{3n+1} = \sqrt{n}$, $a_{3n+2} = 1/(\sqrt{n+2})$.

4. Show that functions $f(z) = |z|$ and $f(z) = \operatorname{Re} z$ are nowhere differentiable on $\mathbb{C}$ though they are continuous on $\mathbb{C}$.

5. Show that there cannot exist a function analytic on an open set $G \subset \mathbb{C}$ with real part $x - 2y^2$.

6. Find all zeros of $1 - \cos z$.

7. Map conformally the infinite strip $\{z : |\operatorname{Re}(z-1)| < \pi/2\}$ onto the open disk.

8. Evaluate the following integrals:

$$\int_\gamma \frac{dz}{z^2+1}$$

where $\gamma(t) = 2e^{it}$, $0 \le t \le 2\pi$; and

$$\frac{1}{i}\int_\gamma \frac{z^m \cos \pi z}{\sin \pi z}\, dz$$

where $\gamma(t) = (n + 1/2)e^{it}$, $0 \le t \le 2\pi$, n and m are positive integers.

9. Find the Laurent expansion of $1/z(z-1)$ in $\{z : 0 < |z| < 1\}$ and also the Laurent expansion of the same function in $\{z : 0 < |z-1| < 1\}$.

10. Show that

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\, dx = -\frac{\pi}{4}.$$