SAMPLE **SAMPLE**

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Comprehensive MS Math (and PhD qualifying) Exam: Mathematical Physics

Rules. Complete two problems of your choice.

(1) Prove the Fourier integral representation of the Dirac $\delta-$function:

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixt}dt.$$

(2) Show that the spectrum $\sigma(A)$ of the operator $A = -\frac{d}{dx}(1-x^2)\frac{d}{dx}$ on $L_2(-1,1)$ is discrete and $\sigma(A) = \{m(m+1)\}_{m=0}^{\infty}$.

(3) Show that the Fourier transform defines a unitary operator on $L_2(-\infty,\infty)$. Plancherel's theorem.

(4) Solve the initial value Dirichlet problem for the free wave equation on the interval $(0,\pi)$ by the Fourier method:

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x). \end{cases}$$

(5) Prove that the solution of the Dirichlet problem for the Laplace equation on the unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| = 1\}$:

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\mathbf{D}} = \varphi(\theta) \end{cases}$$

can be represented by Poisson's formula

$$u(x,y) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\varphi(\theta)\,d\theta, z = x+iy.$$

(6) Find the expression for the Green's function of the Schrodinger operator $A = -d^2/dx^2 + q(x)$ on $L_2(a,b)$ with the Dirichlet boundary conditions $u(a) = 0 = u(b)$.