Complex Analysis
Comprehensive Exam Spring 2022

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Complete eight (8) of the following ten (10) problems. Paper and pencil only. To get full credit you must show every essential step of your arguments.

1. Show that
$$\sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}$$
and use it to derive the identity
$$1/2 + \cos\theta + \cos 2\theta + ... + \cos n\theta = \frac{\sin(n+1/2)\theta}{2\sin(\theta/2)}.$$

2. Show that for any $z_1, z_2 \in \mathbb{C}$
$$||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|.$$

3. (a) Give the definition of an analytic function. (b) State the Cauchy-Riemann conditions (equations). (c) Prove that if f is analytic in a domain E then its real and imaginary parts satisfy the Cauchy-Riemann conditions (equations) in E.

4. Define the function z^α ($\alpha \in (0,1)$) to be analytic in $\mathbb{C} \setminus [0,\infty)$. Compute the jump of z^α across $(0,\infty)$ (i.e. the difference $(x+i0)^\alpha - (x-i0)^\alpha$, $x > 0$).

5. Use a known Taylor series to find the Laurent series for $\dfrac{5z}{(3z-1)(2z+1)}$ in the annulus $1/3 < |z| < 1/2$.

6. State and prove the Cauchy integral formula (assume the Cauchy theorem).

7. State the Rouche theorem and use it to show that all poles of $f(z) = (z^4 + 6z + 3)^{-2}$ lie inside the circle $|z| = 2$.

8. Use contour integration to show that for any $0 \le r < 1$
$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{|r-e^{i\theta}|^2} d\theta = 1.$$

9. Evaluate
$$\int_{-\infty}^{\infty} \frac{\sin \pi x}{x^2 + 2x + 4} dx.$$

10. Prove that for any $a > 0$
$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon^+} \frac{e^{iaz}}{z} dz = i\pi,$$
where $C_\varepsilon^+ = \{z : \operatorname{Im} z \ge 0, |z| = \varepsilon\}$ with the initial point ε and the terminal point $-\varepsilon$.